Post Correspondence Problem

An instance of PCP is called a correspondence system and consists of a set of pairs \((\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots (\alpha_n, \beta_n)\), where the \(\alpha_i\)'s and \(\beta_i\)'s are nonnull strings over an alphabet \(\Sigma\).

The problem asks, for a given set of this form, whether there is a sequence of one or more integers \(i_1, i_2, \ldots, i_k\), each \(i_j\) satisfying \(1 \leq i_j \leq n\) and the \(i_j\)'s not necessarily distinct, so that \(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k} = \beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_k}\). If such a sequence exists, the instance is a yes-instance and the sequence is a solution for the instance.

PCP

Think first of \(n\) distinct groups of dominoes, where each domino from the \(i^{th}\) group has the string \(\alpha_i\) on the top half and the string \(\beta_i\) on the bottom half.

Think of an unlimited number of identical dominoes in each group.
PCP

- Consider the correspondence system described by the following diagram.

<table>
<thead>
<tr>
<th>10</th>
<th>01</th>
<th>0</th>
<th>100</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>100</td>
<td>0</td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

- Must use the first domino for any solution

<table>
<thead>
<tr>
<th>10</th>
<th>1</th>
<th>01</th>
<th>0</th>
<th>100</th>
<th>100</th>
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<th>100</th>
</tr>
</thead>
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<td>101</td>
<td>010</td>
<td>100</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Modified PCP

- The same as PCP except that the solution for the instance is required to begin with domino 1.
  - A solution consists of a sequence of zero or more integers $i_2, i_3, ..., i_k$ so that $\alpha_1, \alpha_{i_2}, ..., \alpha_{i_k} = \beta_1, \beta_{i_2}, ..., \beta_{i_k}$
MPCP ≤ PCP

- MPCP is reducible to PCP.
- The $\text{HALT}_{TM}$ problem is reducible to MPCP; therefore MPCP is undecideable.
- Based upon these two facts,
  - If MPCP is undecidable, then PCP is also undecideable.

Example

- Show that the PCP is decidable over the unary alphabet $\Sigma = \{1\}$.
  - The PCP over a unary alphabet is decidable, therefore we can describe a TM $M$ that decides unary PCP.
Example

- In the silly PCP, SPCP, in each pair the top string has the same length as the bottom string. Show that the SPCP is decidable.

Example

- Let AMBIG\textsubscript{CFG} = \{<G> | G is an ambiguous CFG\}. Show that AMBIG\textsubscript{CFG} is undecidable.
  - Hint: Use a reduction from PCP. Given an instance
    \[ P = \left\{ \begin{bmatrix} \ell_1 \\ b_1 \\ \\ \\ \ell_k \\ b_k \end{bmatrix}, \ldots, \begin{bmatrix} \ell_1 \\ b_1 \\ \\ \\ \ell_k \\ b_k \end{bmatrix} \right\}, \]
    of the PCP, construct a CFG \( G \) with the rules
    
    \[
    \begin{align*}
    S &\rightarrow T \mid B \\
    T &\rightarrow t_1 T a_1 \mid \ldots \mid t_1 a_{k-1} t_1 a_k \mid \ldots \mid t_k a_k \\
    B &\rightarrow b_1 B a_1 \mid \ldots \mid b_1 B a_k \mid \ldots \mid b_k a_k \\
    \end{align*}
    \]
    Where \( a_1, \ldots, a_k \) are new terminal symbols.)