Reducing a Problem to Another

Definition: Given problems A and B, if problem A reduces to problem B, then the solution to problem B can be used to solve problem A.

- When A is reducible to B, solving A cannot be harder than solving B since a solution to B provides a solution to A.

The Halting Problem

- The fact that the Halting problem is undecidable can be used to prove that other decision problems are undecidable.
  - The Halting Problem: A decision problem can be decided by a Turing machine (accepted by a TM that terminates on every input) iff the problem can be solved by a program that halts on all input strings.
    - The problem of determining whether a TM accepts a given input string.
  - For example the “dead-code problem” is undecidable.
The Dead-Code Problem

Dead-Code Problem: Given a program $P$, and input $x$, and a line $n$ in $P$. Does $P$ on input $x$ ever reach line $n$?

Proof:
Suppose for contradiction that the dead-code problem is decidable.

We will use the solution from the dead-code problem to solve the halting problem.

The Dead-Code Problem

The halting problem reduces to the dead-code problem.

Look at an arbitrary C++ Program $P$ and input $x$. We want to decide if $P$ on input $x$ gets into an infinite loop or not.
The Dead-Code Problem

P halts iff P gets to the end of the program. Suppose the last line of the program is line k. Then P halts on input x iff P on input x reaches line k. By our assumption, this is decidable. It follows that the halting problem is decidable, but this is a contradiction.

If follows that our assumption that the dead-code program is decidable is incorrect. So, we have shown that the dead-code program is undecidable.

Undecidable TM Problems

- The following problems (and many others) for TMs are all undecidable:
  1. Given a TM M and an input x, does M terminate on input x? (HALT$_{TM}$)
  2. Given a TM M and an input x, is x ∈ L(M)?
  3. Given a TM M, is ε ∈ L(M)?
  4. Given a TM M, is L(M) = ∅? (E$_{TM}$)
### Undecidable TM Problems

5. Given two TMs $M_1$ and $M_2$, is $L(M_1) = L(M_2)$? (EQ$_{TM}$)
6. Given a TM $M$, does TM $M$ have an equivalent FA? (REGULAR$_{TM}$)
7. Given a TM $M$, is $L(M)$ nonempty?
8. Given a TM $M$, with input alphabet $\Sigma$, is $L(M) = \Sigma^*$?

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The first problem is the halting problem from the previous dead-code problem discussion. It has simply been reformulated for TMs.

Use the fact that problem 1 is undecidable to show that the other problems are undecidable as well.
Undecidable TM Problems

Problem 2: Given a TM $M$ and an input $x$, is $x \in L(M)$?
- Assume that this problem is decidable and that the Halting$_{TM}$ problem can be reduced to this problem, for purposes of obtaining a contradiction.
- Assume that TM $M$ decides HALT$_{TM}$. Construct TM $M_1$ to decide problem 2 that incorporates TM $M$. If $M$ decides HALT$_{TM}$, then $M_1$ decides this problem. However, HALT$_{TM}$ is undecidable, therefore this problem must also be undecidable.

Problem 3: Given a TM $M$, is $\varepsilon \in L(M)$?
- Similar to the previous proofs, the HALT$_{TM}$ problem can be reduced to this problem and a contradiction can be found.
Undecidable TM Problems

- $E_{\text{TM}}$: Given a TM $M$, is $L(M) = \emptyset$?
  - Problems 2 and 3 can be reduced to $E_{\text{TM}}$ to prove that this problem is undecidable.

Assume TM $M$ and string $w$

- Construct $M_1$
  - Hypothetical Halting TM
    - Yes, $M_1(w) = \emptyset$
    - No, $M_1(w) \neq \emptyset$
  - No, $M$ rejects $w$
  - Yes, $M$ accepts $w$

Undecidable TM Problems

- $E_{\text{EQ}}$: Given two TMs $M_1$ and $M_2$, is $L(M_1) = L(M_2)$?
  - The reduction proof requires reducing $E_{\text{TM}}$ to $E_{\text{EQ}}$ which again provides a contradiction and demonstrates that this problem is undecidable.

Assume TM $M_1$ represents $E_{\text{EQ}}$

- Convert $M_1$
  - Hypothetical Halting TM
    - Yes, $L(M_1) = L(M_2)$
    - No, $L(M_1) \neq L(M_2)$
  - Yes, $L(M) = \emptyset$
  - No, $L(M) \neq \emptyset$
Rice’s Theorem

- Given a TM $M$, is $\varepsilon \in L(M)$?
  - This particular problem can be stated more generally as:
    Given a TM $M$, does $L(M)$ have property $P$?

Rice’s Theorem

**Rice’s Theorem**: If $P$ is a property of languages that is satisfied by some but not all recursively enumerable languages, then the decision problem

$D_P$: Given a TM $M$, does $L(M)$ have property $P$?

is undecidable.
Rice’s Theorem

Proof outline:
- Assume that $P$ is a nontrivial language property in the sense of the theorem. We need to show that $D_P$ is reducible to another problem that is known to be undecidable.
  - A property of a RE language is a set of languages.
  - A property is trivial if it is either empty or is all RE languages. Otherwise it is nontrivial.
- $D_P$ is reducible to problem 3, that we have already shown is undecidable. We create a new TM that represents $D_P$ and provides the same answers as for the TM representing problem 3. This can then be used to show that $D_P$ is undecidable.

Rice’s Theorem and Undecidability

A number of undecidable problems follow immediately from Rice’s Theorem.
- Problem 7: Given a TM $M$, is $L(M)$ nonempty?
- Problem 8: Given a TM $M$, with input alphabet $\Sigma$, is $L(M) = \Sigma^*$?
- Problem 9: Given a TM $M$, does $M$ accept at least two strings?
- Problem 10: Given a TM $M$, is the language accepted by $M$ finite?
- Problem 11: Given a TM $M$, is the language accepted by $M$ regular?
- Problem 12: Given a TM $M$, is the language accepted by $M$ recursive?
Example

- Consider the problem of determining whether a two-tape Turning Machine ever writes a nonblank symbol on its second tape when it is run on input $w$. Formulate this problem as a language, and show that it is undecidable.
  - Let $B = \{<M, w> \mid M$ is a two-tape TM that writes a nonblank symbol on its second tape when it is run on $w\}$.
  - Show that $A_{TM}$ reduces to $B$. Assume for purposes of contradiction that TM $R$ decides $B$. Construct TM $S$ that uses $R$ to decide $A_{TM}$.

Example

- Consider the problem of determining whether a two-tape Turning Machine ever writes a nonblank symbol on its second tape during the course of its computation on any input string. Formulate this problem as a language, and show that it is undecidable.
  - Let $C = \{<M> \mid M$ is a two-tape TM that writes a nonblank symbol on its second tape when it is run on some input\}$.
  - Show that $A_{TM}$ reduces to $C$. Assume for purposes of contradiction that TM $R$ decides $C$. Construct TM $S$ that uses $R$ to decide $A_{TM}$. 
Example

- Show that the following problem is undecidable. Given a TM $T$ and a string $w$, does $T$ loop forever?

Example

- Show that the following problem is undecidable. Given a TM $T$, does it accept the string $\varepsilon$ in an even number of moves?
  - We need to reduce the problem $P$: Given a TM $M$, is $\varepsilon \in L(M)$? to this problem.
Example

- Use Rice’s Theorem to prove the undecidability of each of INFINITE$_{TM} = \{<M> \mid M$ is a TM and $L(M)$ is an infinite language}.
  - Rice’s Theorem: If $P$ is a property of languages that is satisfied by some but not all recursively enumerable languages, then the decision problem $D_P$: Given a TM $M$, does $L(M)$ have property $P$? is undecidable.
  - INFINITE$_{TM}$ is a language of TM descriptions.
    - It satisfies the two conditions of Rice’s Theorem.

Computation History

- Definition: The computation history for a TM on an input is the sequence of configurations that the machine goes through as it processes the input.
A Linear Bounded Automaton (LBA) is a TM with a limited amount of memory.

- LBAs accept Context Sensitive Languages.

LBAs can be used to decide the following problems.

- \( A_{\text{DFA}} = \{(B, w) \mid B \text{ is a DFA that accepts input string } w\} \)
- \( A_{\text{CFG}} = \{(G, w) \mid G \text{ is a CFG that generates string } w\} \)
- \( E_{\text{DFA}} = \{(A) \mid A \text{ is a DFA and } L(A) = \emptyset\} \)
- \( E_{\text{CFG}} = \{(G) \mid G \text{ is a CFG and } L(G) = \emptyset\} \)
- Every CFL can be decided by an LBA.
Linear Bounded Automaton

- Given a LBA $M$ and a string $w$, does $M$ accept $w$?
  - This is a decidable problem.
    - The proof requires the lemma: Let $M$ be a LBA with $q$ states and $g$ symbols in the tape alphabet. There are exactly $q^n g^n n$ distinct configurations of $M$ for a tape of length $n$.
  - Recall that this problem is undecidable for TMs.
- However, $E_{LBA} = \{ (M) \mid M$ is a LBA and $L(M) = \emptyset \}$ is undecidable.

Undecidable Problems for CFLs

- Some undecidable problems for CFLs are:
  1. Given a CFG $G$, is $L(G) = \Sigma^*$?
  2. Given two CFGs $G_1$ and $G_2$, is $L(G_1) = L(G_2)$?
  3. Given two CFGs $G_1$ and $G_2$, is $L(G_1) \cap L(G_2) = \emptyset$?
  4. Given a CFG $G$, is $G$ ambiguous?

- The PDA analogs of problems 1, 2, and 3 are also undecidable.
Example

- Show that the following is undecidable for CFGs. Given two CFGs $G_1$ and $G_2$, is $L(G_1) = L(G_2)$?
  - The problem Given a CFG $G$, is $L(G) = \Sigma^*$? is reducible to this problem.

Example

- Show that the following is undecidable for CFGs. Given two CFGs $G_1$ and $G_2$, is $L(G_1) \subseteq L(G_2)$?
  - The problem Given a CFG $G$, is $L(G) = \Sigma^*$? is reducible to this problem.
Example

- Show that the following is undecidable for CFGs. Given a CFG $G$, is $L(G)'$ finite?
  - We first have to show that Given a TM $T$, is $L(T)$ finite? is undecidable.
  - Then use reduce that solution to the problem above.